Mathematical properties and physical meaning of the gravity gradient tensor eigenvalues

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SUMMARY

The eigenvalues of the gravity gradient tensor can be expressed as functions of two parameters: a magnitude and a phase. The decomposition gives physical meaning to the eigenvalues: the magnitude measures the amount of curvature and the phase is related to the type of source. Their understanding allows to propose new quantities to interpret. As an example, a modified phase eigenvalue offers the interpreter an enhanced version of the vertical gravity gradient which is demonstrated with model data and applied to FALCON airborne gravity gradiometer data from the Perth Basin, Australia

Key words: Gravity, gradient, tensor, eigenvalues.

INTRODUCTION

The gravity gradient tensor can be represented by a 3×3 real traceless symmetric matrix in source free regions. The method for obtaining the eigenvalues of a general 3×3 general matrix involves finding the roots of a third order polynomial and has been known for a long time. Pedersen and Rasmussen (1990) exhibit the solutions for our case. Interpreting the eigenvalues has proven to be an extremely difficult task. By identifying the gravity gradient tensor with the tidal tensor (Cevallos, 2016), one can consider the following scenario: In the absence of external forces, a water droplet in outer space is shaped like a sphere; when subjected to a gravitational field gradient the droplet will be shaped like an ellipsoid. How much the droplet stretches and compresses along each axis of the ellipsoid is given by the eigenvalues, the orientation of the axes is given by the eigenvectors.

The purpose of this paper is to examine the mathematical properties, to find physical meaning of the eigenvalues of the gravity gradient tensor and to show an example of how this knowledge can be used in interpretation. There is very large body of work on eigenvalues and eigenvectors of tensors, in particular tensors represented by real symmetric matrices: Stress, strain, diffusion, conductivity, etc. For example, an instance of this, is the normalized source strength calculated from the eigenvalues of the magnetic gradient tensor (Clark 2012). It is always possible to write a tensor as the sum of a symmetric and an antisymmetric part. A real symmetric tensor can always be transformed to a real diagonal tensor expressed in a particular coordinate system formed by principal directions, principal components or eigendirections as they might be called. There is also a large body of work (a fair amount of it proprietary) on the acquisition and processing of airborne gravity gradient data; these procedures change with different acquisition systems, have different sensitivities, parameters, noise characteristics, interpolation and filtering schemes. Reviewing tensor eigenvalues and eigenvector theory or acquisition and processing techniques is beyond the scope of this paper.

I will present the basic, required definitions of the gravity gradient tensor eigenvalues and invariants as proposed by Pedersen and Rasmussen (1990) and illustrate their main properties when applied to synthetic data and field data.

THEORY

Define the potential field as a function of position $\Phi = \Phi(x, y, z)$ in the three orthogonal directions x, y, and z; the gravity field is then represented by a three component vector Φk: (k = x, y, z) and the gravity gradient tensor Ψ by a matrix Φij: (i, j = x, y, z) where the subscripts indicate partial differentiation with respect to the subscripted variable.

$$\Psi = \begin{pmatrix} \Phi xx & \Phi xy & \Phi xz \\ \Phi yx & \Phi yy & \Phi yz \\ \Phi zx & \Phi zy & \Phi zz \end{pmatrix} \qquad (1)$$

The gravity gradient tensor is symmetric, that is, $\Phi ij = \Phi ji$, and its trace is zero in source-free regions, that is, $\Phi xx + \Phi yy + \Phi zz = 0$. The characteristic equation for determining the eigenvalues λi (i = 1, 2, 3) is

$$\lambda^3 + I_1 \lambda - I_2 = 0, \qquad (2)$$

where

$$I_1 = \Phi xx\, \Phi yy + \Phi yy\, \Phi zz + \Phi xx\, \Phi zz - \Phi^2 xy - \Phi^2 yz - \Phi^2 xz, \qquad (3)$$

$$I_2 = \Phi xx\, (\Phi yy\Phi zz - \Phi^2 yz) + \Phi xy\, (\Phi yz\Phi xz - \Phi xy\Phi zz) + \Phi xz\, (\Phi xy\Phi yz - \Phi xz\Phi yy), \qquad (4)$$

where I_2 is $\det(\Psi)$, are invariants under any coordinate transformation and $I_1 \leq 0$.

Let $Q = - I_1/3$ and $R = I_2/2$, then Equation 2 has the real roots (Nickalls, 1993; Kindlmann, 2004)

$$\lambda_1 = 2\sqrt{Q}\cos\Theta, \tag{5}$$
$$\lambda_2 = 2\sqrt{Q}\cos(\Theta - 2\pi/3), \tag{6}$$
$$\lambda_3 = 2\sqrt{Q}\cos(\Theta + 2\pi/3), \tag{7}$$

where

$$\Theta = (1/3)\arccos(R/\sqrt{(Q^3)}), \tag{8}$$

and

$$\lambda_1 \geq \lambda_2 \geq \lambda_3. \tag{9}$$

From Equation 8 it can be observed that the range of Θ is bounded, as $R/\sqrt{(Q^3)}$ varies in the interval $[-1, 1]$ Θ varies in the interval $[\pi/3, 0]$ (Kindlmann, 2004).

The eigenvalues can be displayed using the so called eigenvalue wheel used by Nickalls (1993) and Kindlmann (2004) (Figure 1). The wheel allows visualization of the magnitude ($2\sqrt{Q}$) and the phase angles of the three eigenvalues. The terms magnitude and phase in the context of the gravity gradient tensor or tidal tensor have been used before by FitzGerald et al. (2007) and FitzGerald et al. (2009), however the meaning of the terms in their work is different as they refer to eigenvalues and eigenvectors linked via a quaternion representation. Here, I am referring exclusively to the eigenvalues.

In practice, the eigenvalues and, to a lesser degree, the invariants have proven to be difficult to interpret, largely due to the fact that no physical meaning has been attached to them. There are many ways to parametrize three eigenvalues of zero sum in terms of two invariants, the way shown in Equations 5, 6 and 7 enables us to assign physical meaning to the eigenvalues through the understanding of $2\sqrt{Q}$ and Θ.

In general relativity, one of the most basic curvature invariants is the Kretschmann scalar (Stephani et al., 2009). The Kretschmann scalar is defined as the sum of squares of all the components of a given tensor. The Riemann curvature tensor (Misner et al., 1973; Schutz, 1985) is a four-index tensor that is extensively used in general relativity. For the special case when the Riemann curvature tensor reduces to the Newtonian tidal tensor (i.e. the gravity gradient tensor), the Kretschmann scalar is

$$K = - 2I_1 = 6Q. \tag{10}$$

As the Kretschmann scalar is proportional to the amount of curvature, it is used in gravitational theory to calculate where the gravitational field is singular (Gkigkitzis et al., 2014; Abdelqader, 2013). In our case, it is the 3-D equivalent of Casorati's curvature (Marussi, 1979) that is used to calculate the curvature of surfaces such that when the surface is a flat plane it is equal to zero. Mathematically, the Kretschmann scalar is equal to the square of the Frobenius norm (Golub and van Loan, 1996).

Pedersen and Rasmussen (1990) define the dimensionless quantity I as

$$0 \leq I = - [(I_2/2)^2] / [(I_1/3)^3] \leq 1. \tag{11}$$

They point out, that when I is equal to zero it is an indicator of two dimensionality of the source, although it is not a sufficient condition as there are three dimensional cases where it is also equal to zero. They also show that for a point source I is equal to one.

Substitution of Q and R in Equation 11 yields

$$I = (R/\sqrt{(Q^3)})^2. \tag{12}$$

Equations 5 to 12 give intuitive meaning to the eigenvalues of Ψ. For example, we can now think of the eigenvalues as projections derived from the same magnitude (or length) with different phases. It also means that we do not need to look at the three eigenvalues (λ_i) (of which only two are independent), but rather at the quantities $2\sqrt{Q}$ and Θ. Examination of the magnitude $2\sqrt{Q}$ and the phase Θ and their spatial behaviors should yield source position curvature information (horizontal location, edges, etc.) in one case and indications or clues of source nature (type of contrast, shapes, etc.) in the other.

It is interesting to note that expanding invariant I_2 such that

$$I_2 = \det(\Psi) = \lambda_1\lambda_2\lambda_3 = (2\sqrt{Q})^3(\cos\Theta)[\cos(\Theta - 2\pi/3)][\cos(\Theta + 2\pi/3)], \tag{13}$$

and using the trigonometric identity

$$\cos 3\Theta = 4\cos^3\Theta - 3\cos\Theta, \tag{14}$$

yields

$$I_2 = [(2\sqrt{Q})^3/4]\cos(3\Theta). \tag{15}$$

Mathematically, Equations 13 and 15 show that except for the ¼ factor in Equation 15 the multiplication of the eigenvalues that yields I_2 is similar to a complex number multiplication (product of magnitudes and addition of phases) (Figure 1).

MODELING

Basic visualization of the horizontal variations of $2\sqrt{Q}$ and Θ can be achieved using the synthetic model used in Cevallos et al. (2013). In the model, two 1 km³ model cubes were assigned density contrasts of + (left body) and − (right body) 1.0×10^3 kg/m³ respectively in a 2.67×10^3 kg/m³ background density. The cubes are separated by 1 km and have a depth to top of 100 m.

The eigenvalues and Φ_{zz} for comparison are shown in Figures 2, 3, 4 and 5. They show the typical behavior of what is known as the positive asymmetric eigenvalue λ_1, the symmetric eigenvalue λ_2 and the negative asymmetric eigenvalue λ_3 (Dransfield; 1994) and Φ_{zz}. The magnitude $2\sqrt{Q}$ (Figure 6) shows the expected symmetric behavior, good positioning of the sources and a good definition of the boundaries. The phase Θ (Figure 7) shows the expected anti symmetric behavior and a complex point source behavior away from

the bodies and in arcs that go from corner to corner avoiding the edges of the bodies. It can now be seen why the eigenvalues are so difficult to interpret: The phase Θ shows a very complex spatial behavior even for a simple case like the one modeled here. This fact discourages the use of Θ in interpretation. What would be desirable is that instead of having the magnitude $2\sqrt{Q}$ multiplied by $\cos\Theta$, $\cos(\Theta - 2\pi/3)$, and $\cos(\Theta + 2\pi/3)$ to generate the eigenvalues, another quantity could be used such that interpretation becomes easier. A natural candidate is the shape index (SI) (Koenderink and van Doorn, 1992; Roberts, 2001; Cevallos et al., 2013; Cevallos, 2014):

$$SI = (2/\pi)\arctan\{\Phi zz/[(\Phi xx - \Phi yy)^2 + 4\Phi xy^2]^{\frac{1}{2}}\}. \tag{16}$$

The shape index is a smooth continuous function that like the cosine function has values defined in the range [−1, 1]. It is related to the type of source as it describes the shape of the equipotential surfaces that the sources generate. A modified phase eigenvalue can then be defined as:

$$\lambda si = 2\sqrt{Q}\ SI. \tag{17}$$

The shape index and the modified phase eigenvalue λsi of the synthetic cubes are shown in Figures 8 and 9. The preservation of the sign of the density contrast is clearly illustrated in Figure 9 compared with the magnitude of $2\sqrt{Q}$ illustrated in Figure 6.
I have found the modified phase eigenvalue useful in interpretation as it gives a slightly sharper definition of the edges of the sources than the eigenvalues or the vertical gravity gradient Φzz.
As the gravity gradient tensor Ψ is symmetric and is traceless, it has five independent quantities to interpret. Interpreting two quantities: the invariants, the eigenvalues, the modulus and the phase, or the modulus and the shape index cannot give a complete description of sources. There are three more quantities to consider, the ones associated with the three eigenvectors, namely three angles.

APPLICATION

The modified phase eigenvalue was calculated for the FALCON® AGG data from the central part of the Perth Basin, Western Australia (Kovac et al., 2016) (Figure 17).
The Perth Basin is part of the continental margin of south-western Australia. It is bordered by the Archean Yilgarn Craton to the east, the Southern Carnarvon Basin to the north, the Bight Basin to the southeast and oceanic crust of the Indian and Southern Oceans to the west and south, respectively. The creation of the basin is related to the breakup of Gondwanaland and the formation of the Indian Ocean. The basin contains up to 15 km of mid-Carboniferous to Lower Cretaceous sedimentary rocks (Thomas, 2014), which trace a long-lived, mostly non-marine depositional environment with occasional marine incursions (FrOG Tech, 2006; Playford et al., 1976). The structural architecture of the Perth Basin is the result of oblique rifting during the Permian, Late Triassic to Early Jurassic and Middle Jurassic to Early Cretaceous, superimposed over pre-existing basement terrains. Metamorphic rocks of the Mesoproterozoic to Neoproterozoic Pinjarra Orogeny, which are exposed in the Leeuwin, Mullingarra, and Notthampton Inliers, determine the geological basement of the Perth Basin. The survey was flown east-west with a nominal flying height of 100 m with 1000 m line spacing, using a flight line to tie line ratio of 10:1. For interpretation purposes, gravity data was terrain corrected using density values of 1.4×10^3 kg/m^3 in the western part, and 2.0×10^3 kg/m^3 in the eastern part (Figure 17). As the eastern part contains the most promising identified leads and gas discoveries all the following images were calculated using 2.0×10^3 kg/m^3 terrain corrected data. The survey identified large structural leads as the target of future gas exploration including infill 2D seismic acquisition and that the prospectivity of some of the leads increased significantly with gas discovery wells (Kovac et al., 2016).

Figures 18, 19, 20 and 21 (all figures have illuminated shading from the northeast with an elevation of 45 degrees) show the vertical gravity gradient Φzz, magnitude $2\sqrt{Q}$, shape index SI and modified phase eigenvalue λsi. Comparison between the Φzz (Figure 18) and the λsi (Figure 21) images shows two general results and an observation:

- Comparable dynamic ranges.
- λsi is sharper than Φzz, that is, λsi appears to be a subtle enhanced version of Φzz.
- There appear to be artifacts associated with line direction (east west) in the central eastern part of the survey.

To more clearly illustrate the differences between the Φzz and λsi I have zoomed into the central part of the survey for the images in Figures 18 and 21 and put them side by side in Figure 22 (some pixelation can be observed).
As Φzz forms the basis of all gravity gradient interpretations, the second result is significant. It is important to note that the result is not obtained by taking spatial derivatives of gradients, for example, taking the first vertical derivative of Φzz or $2\sqrt{Q}$, a procedure that looks good when applied to with model data, but when applied to real cases it is not satisfactory due to the concurrent enhancement of noise.

The geological significance of the last result is the enhancement of particular trends, for example: north south in the center of the area, northwest southeast in the west of the area and east-west in the center of the area. If these enhanced trends had been available at the time of the interpretation it is unlikely they would have changed the interpretation results, but they would have made them easier to obtain and changed their relative importance.

Why is λsi sharper than Φzz? As all the eigenvalues have the common magnitude term $2\sqrt{Q}$, if a diagonal matrix representing the tensor in an eigenvector coordinate system is rotated to a (x, y, z) coordinate system, all the tensor components Φij (including Φzz) will be linear combinations of the eigenvalues. They will all therefore have the common term $2\sqrt{Q}$ multiplied by coefficients comprising products and sums of sines and cosines of the three rotation angles (yaw, pitch and roll) and the phase Θ. For the Φij in general, this process will alter and most likely diminish the original resolution of $2\sqrt{Q}$.
It is encouraging that a simple scheme of phase substitution yields such model and application results. It is clear there are at least two more schemes to propose and test: other angles as phases and noise mitigated horizontal gradients of the magnitude.

CONCLUSIONS

The concept of tidal tensor gives meaning to the eigenvalues and eigenvectors of the gravity gradient tensor.

The eigenvalues can be expressed as functions of a magnitude and a phase. This decomposition gives particular physical meaning to the eigenvalues: the magnitude measures the amount of curvature and the phase is related to the type of source. The modified phase eigenvalue proposed here, offers the interpreter an enhanced version of the vertical gravity gradient.

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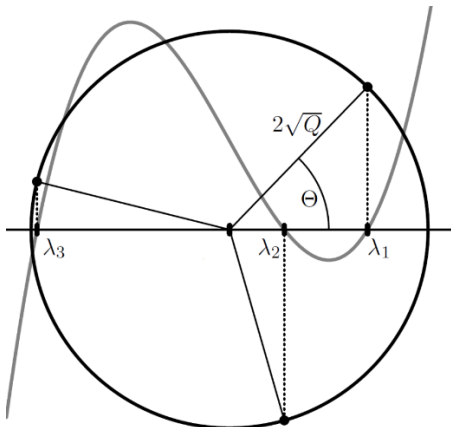
FIGURES

Figure 1. Diagram of cubic characteristic polynomial (in gray), its roots, and the invariants required for determining the roots with Equations 5, 6 and 7 (after Kindlmann, 2004).
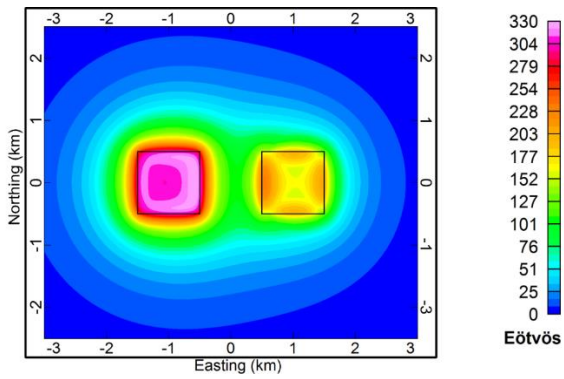
Figure 2. Eigenvalue λ_1, over synthetic model cubes, shown by black outlines (denser body on the left).
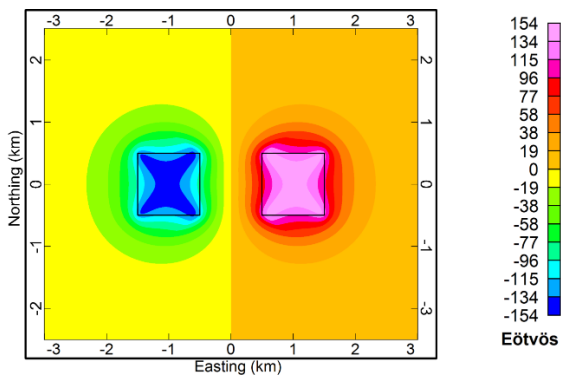

Figure 3. Eigenvalue λ_2, over synthetic model cubes, shown by black outlines (denser body on the left).
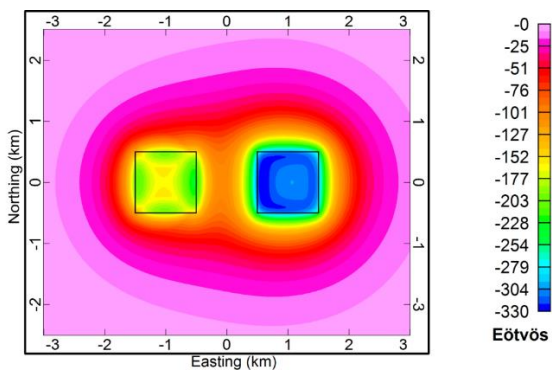

Figure 4. Eigenvalue λ_3, over synthetic model cubes, shown by black outlines (denser body on the left).
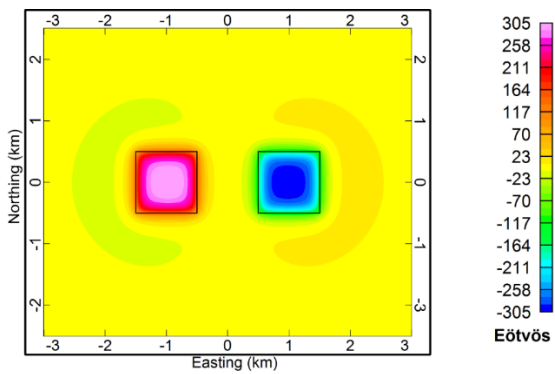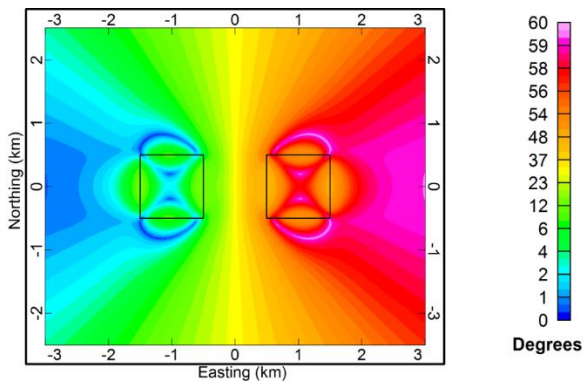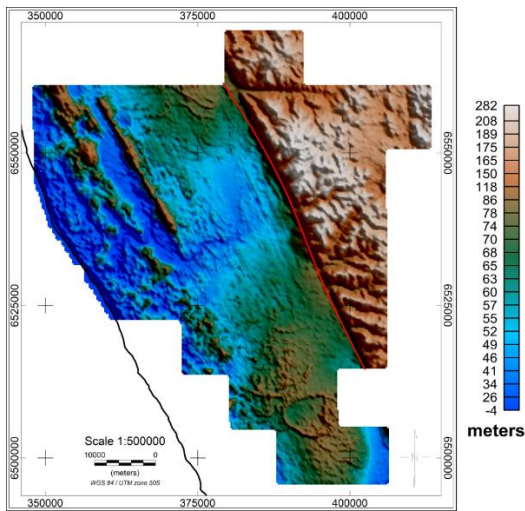

Figure 5. Vertical gravity gradient Φ_{zz}, over synthetic model cubes, shown by black outlines (denser body on the left).

Figure 6. Magnitude 2√Q, over synthetic model cubes, shown by black outlines (denser body on the left).

Figure 7. Phase Θ, over synthetic model cubes, shown by black outlines (denser body on the left).

Figure 8. Shape index SI, over synthetic model cubes, shown by black outlines (after Cevallos et al., 2013) (denser body on the left).

Figure 9. Modified phase eigenvalue λsi, over synthetic model cubes, shown by black outlines (denser body on the left).

Figure 17. Location of the Perth survey area over topography. Coastline in black outline. Terrain corrections used a density value of 1.4×10^3 kg/m^3 to the west of the red outline and a density value of 2.0×10^3 kg/m^3 to the east of the red outline. Sunshaded from the NE with a sun elevation of $45°$.
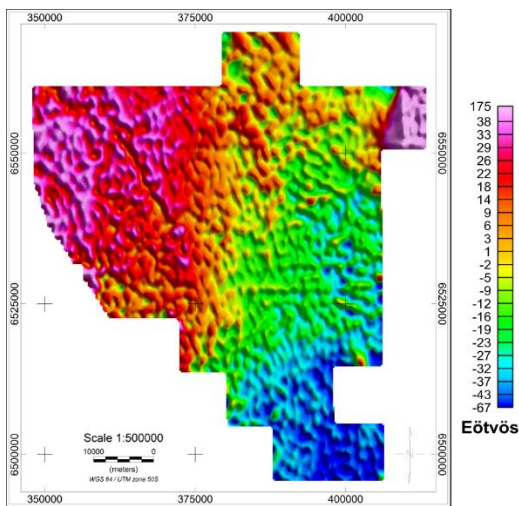

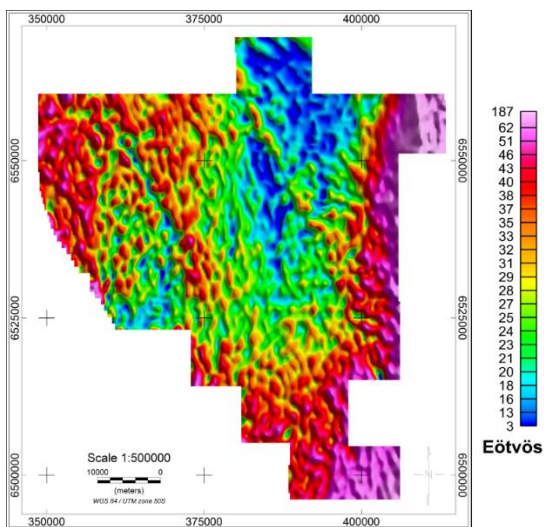**Figure 18. Vertical gravity gradient Φ_{zz}, of survey area. Sunshaded from the NE with a sun elevation of $45°$.**

Figure 19. Magnitude $2\sqrt{Q}$, over survey area. Sunshaded from the NE with a sun elevation of $45°$.
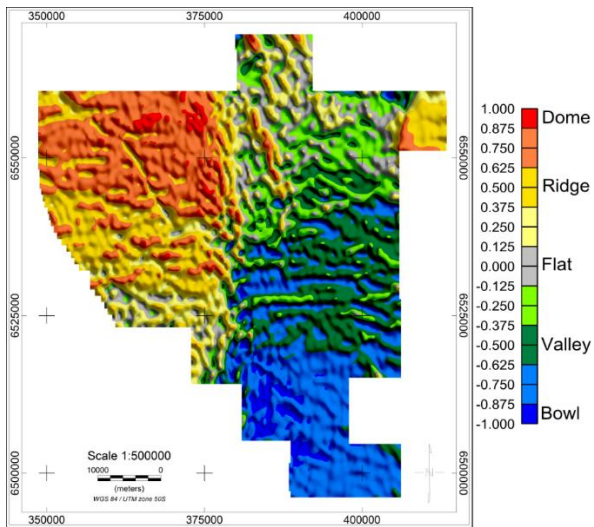
Figure 20. Shape Index SI of equipotential surfaces, over survey area. Sunshaded from the NE with a sun elevation of 45°.
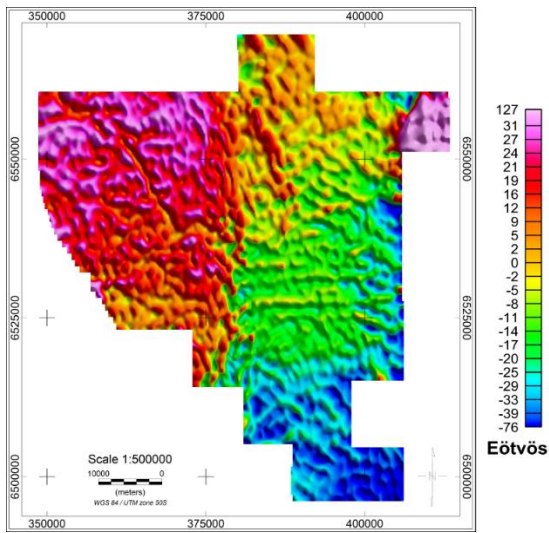

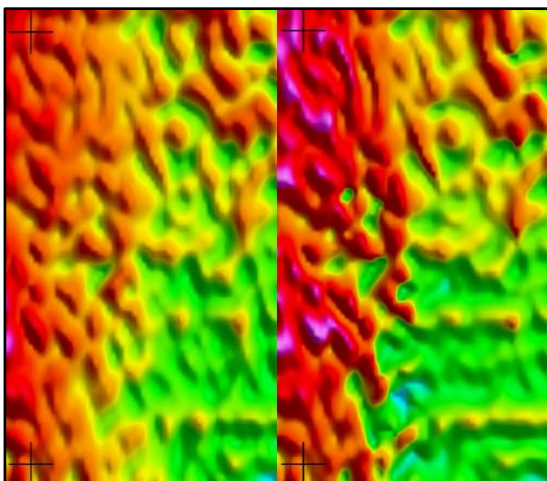**Figure 21. Modified phase eigenvalue λsi, over survey area. Sunshaded from the NE with a sun elevation of 45°.**

Figure 22. Vertical gravity gradient Φzz (left) and modified phase eigenvalue λsi (right), over central survey area. Sunshaded from the NE with a sun elevation of 45°.